On a relationship between the ∗-congruence Sylvester equation and the generalized Sylvester equation

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CCS CONCEPTS

• **Mathematics of computing** → *Computations on matrices*;

KEYWORDS

matrix equation, ∗-congruence Sylvester equation, linear operator

1 INTRODUCTION

In this study, we consider the ∗-congruence Sylvester equation

$$AX + X^{*}B = C, \tag{1}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, and $C \in \mathbb{C}^{m\times m}$, whereas $X \in \mathbb{C}^{n\times m}$ is to be determined. The operator $(\cdot)^{*}$ denotes the conjugate transpose of a matrix and $\mathbb{C}$ denotes the set of all complex numbers. Equation (1) is regarded as an extension of the T-congruence Sylvester equation

$$AX + X^{\mathrm{T}}B = C. \tag{2}$$

Equation (1) appears in palindromic eigenvalue problems [2] arising from some realistic applications (see, e.g., [1]).

Recently, it is shown that the T-congruence Sylvester equation (2) is mathematically equivalent to a generalized Sylvester equation [3, 4]. In other words, the transpose of the unknown matrix X can be removed by appropriate transformations. It is indicated that finding numerical solvers of equation (2) reduces to only finding them of the generalized Sylvester equation.

The key idea of [3, 4] is to vectorize matrices. It is known that a linear system with an $m^2 \times mn$ matrix can be obtained by vectorizing the T-congruence Sylvester equation (2). By applying an appropriate linear operator to the linear system and returning vectors into matrices, the generalized Sylvester equation is obtained.

However, for ∗-congruence Sylvester equation (1), the same approach as in [4] cannot be utilized because of the conjugate of the unknown matrix. To overcome this, we separate the real and imaginary parts of matrices, i.e., we obtain a real linear system with a $2m^2 \times 2mn$ matrix. In this study, we consider applying an appropriate linear operator to the linear system and returning it into a matrix equation which does not include the conjugate and the transpose of the unknown matrix. As a result, we demonstrate that the ∗-congruence Sylvester equation (1) is mathematically equivalent to the generalized Sylvester equation.

2 MAIN RESULTS

Our main results are briefly written as follows:

Theorem 2.1. *Let $m \geq n$. Assume that there exists a matrix $S \in \mathbb{C}^{m\times m}$ such that $B^{\mathrm{T}} = S\bar{A}$ and $\lambda_i\bar{\lambda}_j \neq 1$ for the eigenvalues $\lambda_1, \dots, \lambda_m$ of S, where $\bar{A}$ is the conjugate of A. Then, the ∗-congruence Sylvester equation* (1) *is equivalent to the following generalized Sylvester equation:*

$$AX - B^{*}XS^{\mathrm{T}} = C - (S\bar{C})^{\mathrm{T}}.$$

Theorem 2.2. *Let $m \leq n$. Assume that there exists a matrix $D \in \mathbb{C}^{n\times m}$ such that $I_m = AD$ and $\lambda_i\bar{\lambda}_j \neq 1$ for the eigenvalues $\lambda_1, \dots, \lambda_m$ of $S := B^{\mathrm{T}}\bar{D}$, where I_m is the $m \times m$ identity matrix. Then, the ∗-congruence Sylvester equation* (1) *is equivalent to the following generalized Sylvester equation:*

$$A\hat{X} - B^{*}\hat{X}S^{\mathrm{T}} = C,$$

where $\hat{X}$ satisfies $X = \hat{X} - D\hat{X}^{*}B$.

3 CONCLUSION AND FUTURE WORK

In this study, we showed that the ∗-congruence Sylvester equation is mathematically equivalent to the generalized Sylvester equation. The results were obtained by applying an appropriate linear operator to the equivalent linear system. Our future work will focus on efficient numerical algorithms using our results.

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