

On a Relationship between the *-congruence Sylvester Equation and the Generalized Sylvester Equation

Yuki Satake, Tomohiro Sogabe, Tomoya Kemmochi and Shao-Liang Zhang Nagoya University, Japan

*-congruence Sylvester equation

$$AX + X^*B = C$$

$A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times m}, C \in \mathbb{C}^{m \times m}$: given matrices,
 $X \in \mathbb{C}^{n \times m}$: unknown matrix
 $(\cdot)^*$: conjugate transpose of matrix

Applications :

Palindromic eigenvalue problems [1] arising from

- vibration analysis of fast trains
- simulations of surface acoustic wave filter

- can be regarded as an extension of the T-congruence Sylvester equation

$$AX + X^T B = C$$

$(\cdot)^T$: transpose of matrix

Study for the T-congruence Sylvester equation

The T-congruence Sylvester equation is equivalent to a generalized Sylvester equation under certain conditions [2]

$$AX + X^T B = C$$

\Updownarrow

$$AY - B^T Y S^T = Q$$

When $m \geq n$:

- S satisfies $B^T = SA$
- $\lambda_i \lambda_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k : eigenvalue of S)

When $m \leq n$:

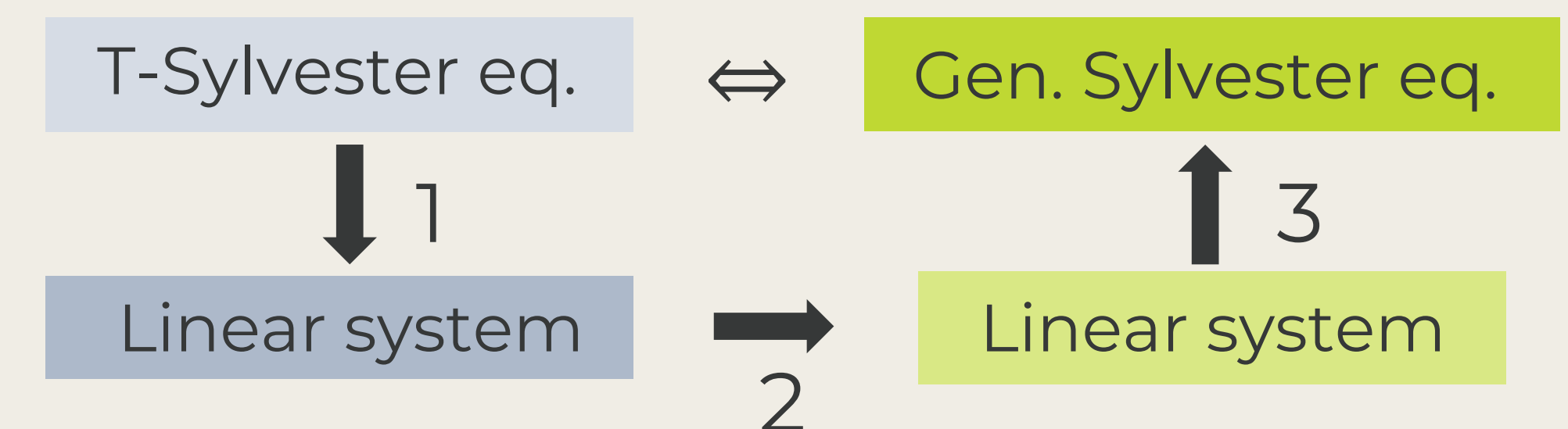
- $S := B^T D$ (D satisfies $I_m = AD$)
- $\lambda_i \lambda_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k : eigenvalue of S)

When $m \geq n, Y := X, Q := C - (SC)^T$
 When $m \leq n, Y := \hat{X}$ (\hat{X} satisfies $X = \hat{X} - D\hat{X}^T B$), $Q := C$

- The generalized Sylvester equation does not contain the transpose of the unknown matrix
 \Rightarrow It may lead to efficient numerical algorithms of the T-Sylvester equation

Approach

1. Apply the vec operator (Vectorization)
2. Transform an equivalent linear system into another linear system
3. Return to matrix equation



Purpose

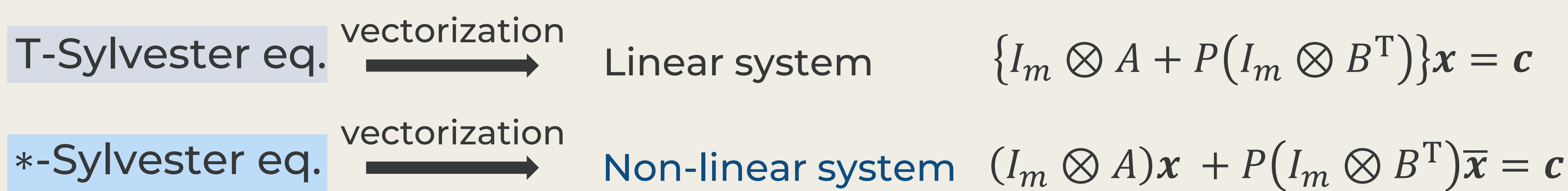
Extend the above previous study to the *-congruence Sylvester equation



Extension to the *-congruence Sylvester equation

Problem:

The same approach cannot be utilized for the *-congruence Sylvester equation



I_m : $m \times m$ identity matrix
 P : permutation matrix
 \otimes : Kronecker product
 $x := \text{vec}(X)$
 $c := \text{vec}(C)$
 $\bar{(\cdot)}$: conjugate
 $\text{vec}(X) := \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{C}^{mn}$

Idea:

Separate the real & imaginary parts respectively

- The non-linear system becomes a real linear system

$$(I_m \otimes A)x + P(I_m \otimes B^T)\bar{x} = c \Leftrightarrow \begin{bmatrix} I_m \otimes A_R + P(I_m \otimes B_R^T) & -\{I_m \otimes A_I - P(I_m \otimes B_I^T)\} \\ I_m \otimes A_I + P(I_m \otimes B_I^T) & I_m \otimes A_R - P(I_m \otimes B_R^T) \end{bmatrix} \begin{bmatrix} x_R \\ x_I \end{bmatrix} = \begin{bmatrix} c_R \\ c_I \end{bmatrix}$$

Main results:

The *-congruence Sylvester equation is equivalent to a generalized Sylvester equation under certain conditions

$$AX + X^*B = C$$

\Updownarrow

$$AY - B^* Y S^T = Q$$

When $m \geq n$:

- S satisfies $B^T = S\bar{A}$
- $\lambda_i \bar{\lambda}_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k : eigenvalue of S)

When $m \leq n$:

- $S := B^T \bar{D}$ (D satisfies $I_m = AD$)
- $\lambda_i \bar{\lambda}_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k : eigenvalue of S)

When $m \geq n, Y := X, Q := C - (S\bar{C})^T$
 When $m \leq n, Y := \hat{X}$ (\hat{X} satisfies $X = \hat{X} - D\hat{X}^* B$), $Q := C$

$(\cdot)_R$: real part, $(\cdot)_I$: imaginary part

Future work

- Develop numerical algorithms for the large *-congruence Sylvester equation using our results

References

1. E. K.-W. Chu et al., Palindromic eigenvalue problems: a brief survey, *Taiwan. J. Math.* 14 (3A) (2010), pp.743–779.
2. Y. Satake et al., Relation between the T-congruence Sylvester equation and the generalized Sylvester equation, *Appl. Math. Lett.* 96 (2019), pp. 7–13.