On a Relationship between the $*$-congruence Sylvester Equation and the Generalized Sylvester Equation

Yuki Satake, Tomohiro Sogabe, Tomoya Kemmochi and Shao-Liang Zhang Nagoya University, Japan

$*$-congruence Sylvester equation

$$AX + X^{*}B = C$$

$A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times m}$: given matrices,
$X \in \mathbb{C}^{n\times m}$: unknown matrix
$(\cdot)^{*}$: **conjugate transpose** of matrix

Applications :

Palindromic eigenvalue problems [1] arising from

- vibration analysis of fast trains
- simulations of surface acoustic wave filter

- can be regarded as an extension of the T-congruence Sylvester equation

$$AX + X^{\mathrm{T}}B = C$$

$(\cdot)^{\mathrm{T}}$: **transpose** of matrix

Study for the T-congruence Sylvester equation

The T-congruence Sylvester equation is equivalent to a generalized Sylvester equation under certain conditions [2]

$$AX + X^{\mathrm{T}}B = C$$

$$\Updownarrow$$

$$AY - B^{\mathrm{T}}YS^{\mathrm{T}} = Q$$

When $m \geq n$:

- S satisfies $B^{\mathrm{T}} = SA$
- $\lambda_i\lambda_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k: eigenvalue of S)

When $m \leq n$:

- $S := B^{\mathrm{T}}D$ (D satisfies $I_m = AD$)
- $\lambda_i\lambda_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k: eigenvalue of S)

When $m \geq n$, $Y := X$, $Q := C - (SC)^{\mathrm{T}}$
When $m \leq n$, $Y := \hat{X}$ ($\hat{X}$ satisfies $X = \hat{X} - D\hat{X}^{\mathrm{T}}B$), $Q := C$

- **The generalized Sylvester equation does not contain the transpose of the unknown matrix**

➡ **It may lead to efficient numerical algorithms of the T-Sylvester equation**

Approach

1. Apply the vec operator (Vectorization)
2. Transform an equivalent linear system into another linear system
3. Return to matrix equation

T-Sylvester eq. ⇔ Gen. Sylvester eq.
↓ 1 ↑ 3
Linear system ➡ 2 Linear system

Purpose

Extend the above previous study to the $*$-congruence Sylvester equation

T-Sylvester eq. ⇔ Gen. Sylvester eq. **Extension?** ➡ $*$-Sylvester eq. ⇔ ????

Extension to the $*$-congruence Sylvester equation

Problem:

The same approach cannot be utilized for the $*$-congruence Sylvester equation

T-Sylvester eq. —vectorization→ Linear system $\{I_m \otimes A + P(I_m \otimes B^{\mathrm{T}})\}\boldsymbol{x} = \boldsymbol{c}$

$*$-Sylvester eq. —vectorization→ Non-linear system $(I_m \otimes A)\boldsymbol{x} + P(I_m \otimes B^{\mathrm{T}})\bar{\boldsymbol{x}} = \boldsymbol{c}$

I_m: $m \times m$ identity matrix
P: permutation matrix
$\otimes$: Kronecker product
$\boldsymbol{x} := \mathrm{vec}(X)$
$\boldsymbol{c} := \mathrm{vec}(C)$
$\overline{(\cdot)}$: conjugate

$$\mathrm{vec}(X) := \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_m \end{bmatrix} \in \mathbb{C}^{mn}$$

Idea:

Separate the real & imaginary parts respectively

- The non-linear system becomes a real linear system

$$(I_m \otimes A)\boldsymbol{x} + P(I_m \otimes B^{\mathrm{T}})\bar{\boldsymbol{x}} = \boldsymbol{c} \quad \Leftrightarrow \quad \begin{bmatrix} I_m\otimes A_{\mathrm{R}} + P(I_m\otimes B_{\mathrm{R}}^{\mathrm{T}}) & -\{I_m\otimes A_{\mathrm{I}} - P(I_m\otimes B_{\mathrm{I}}^{\mathrm{T}})\} \\ I_m\otimes A_{\mathrm{I}} + P(I_m\otimes B_{\mathrm{I}}^{\mathrm{T}}) & I_m\otimes A_{\mathrm{R}} - P(I_m\otimes B_{\mathrm{R}}^{\mathrm{T}}) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\mathrm{R}} \\ \boldsymbol{x}_{\mathrm{I}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_{\mathrm{R}} \\ \boldsymbol{c}_{\mathrm{I}} \end{bmatrix}$$

$(\cdot)_{\mathrm{R}}$: real part, $(\cdot)_{\mathrm{I}}$: imaginary part

Main results:

The $*$-congruence Sylvester equation is equivalent to a generalized Sylvester equation under certain conditions

$$AX + X^{*}B = C$$

$$\Updownarrow$$

$$AY - B^{*}YS^{\mathrm{T}} = Q$$

When $m \geq n$:

- S satisfies $B^{\mathrm{T}} = S\bar{A}$
- $\lambda_i\bar{\lambda}_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k: eigenvalue of S)

When $m \leq n$:

- $S := B^{\mathrm{T}}\bar{D}$ (D satisfies $I_m = AD$)
- $\lambda_i\bar{\lambda}_j \neq 1$ for $\lambda_1, \dots, \lambda_m$ (λ_k: eigenvalue of S)

When $m \geq n$, $Y := X$, $Q := C - (S\bar{C})^{\mathrm{T}}$
When $m \leq n$, $Y := \hat{X}$ ($\hat{X}$ satisfies $X = \hat{X} - D\hat{X}^{*}B$), $Q := C$

Future work

- Develop numerical algorithms for the large $*$-congruence Sylvester equation using our results

References

1. E. K.-W. Chu et al., Palindromic eigenvalue problems: a brief survey, *Taiwan. J. Math.* 14 (3A) (2010), pp.743—779.
2. Y. Satake et al., Relation between the T-congruence Sylvester equation and the generalized Sylvester equation, *Appl. Math. Lett.* 96 (2019), pp. 7—13.

HPC Asia 2020 @Fukuoka, Japan