Cascadic Parareal Method for Explicit Time-Marching Schemes

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Introduction

Several parallel-in-time (PinT) methods such as parareal[3] and multigrid-in-time reduction[1, 2] was proven to be successful on various applications using both implicit and explicit schemes.

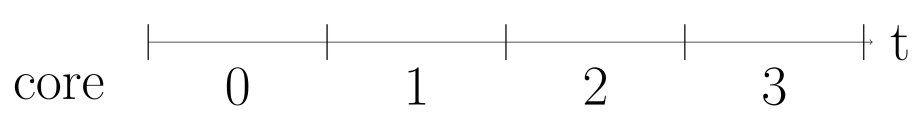
This work propose a parallel-in-time method that minimizes the computation time for **explicit time-marching schemes**.

Challenges

- The Courant-Friedrichs-Lewy condition (CFL condition) is a necessary condition for stability while solving time dependent problems with explicit schemes. This causes coarse operations in PinT methods to fail when time step is too large.
$$C = \frac{u\Delta t}{\Delta x} \leq 1$$
where u is velocity field, Δx is length interval and Δt is time step.
- Parallel-in-time methods are less efficient compare to spatial parallelization, especially for explicit schemes.
- Parallel-in-time methods works poorly on hyperbolic equations and and non-linear flows such as shock wave simulation.

Cascadic Parareal

The whole time line is divided into N segments, which is the number of available cores. Each core only solves in the assigned time segment.



The proposed method constructs multiple coarse layers, each coarser from the other. In each layer, solve the problem with an explicit time-marching scheme $\mathcal{F}$ with different time step.

The difference of the fine and coarse results are then updated sequentially by an explicit scheme on the coarse grid.

$$y^{i+1} = \mathcal{F}(y^i, r\Delta t) + \mathcal{F}(y^{i-1}, \Delta t) - \mathcal{F}(y^{i-1}, r\Delta t)$$

y^i is the result at first time step of core i.

Solve the previous step iteratively using layer $L-1$ to 1 as fine grid until convergence.

In order to prevent violating the CFL condition, we coarsen the x-grid and the time grid at the same time.

$$C' = \frac{v(r\Delta t)}{r\Delta x} = \frac{v\Delta t}{\Delta x} = C$$

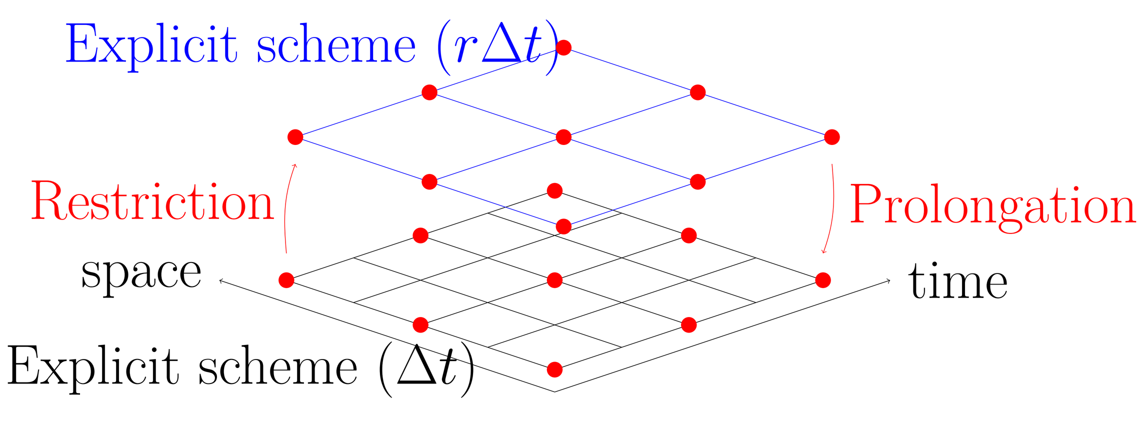


Figure 1:Overview graph of our method

1D Advection

We consider the following advection example with constant velocity and sine wave initial condition.

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}$$

with initial condition:

$$u(x, 0) = \sin^4(\pi x)$$

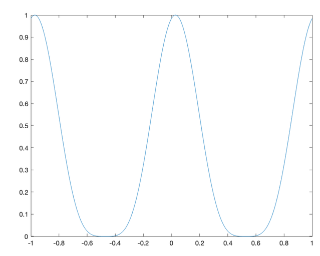


Figure 2:Initial condition

Spatial restriction is defined as follows:

$$y_i = Y_{2i-1}$$

Spatial prolongation is defined as adding the average residual of the neighbor points:

$$Y_{2i} = Y_{2i} + \frac{(Y_{2i-1} - y_i) + (Y_{2i+1} - y_{i+1})}{2}$$

Result (Advection)

This experiment construct a 7 layer parallel-in-time method. In each layer, Lax-Wendroff scheme (explicit) is applied. The problem size is 4096 in space and 8192 in time dimension. This experiment is tested on Oakbridge-CX cluster at the University of Tokyo.

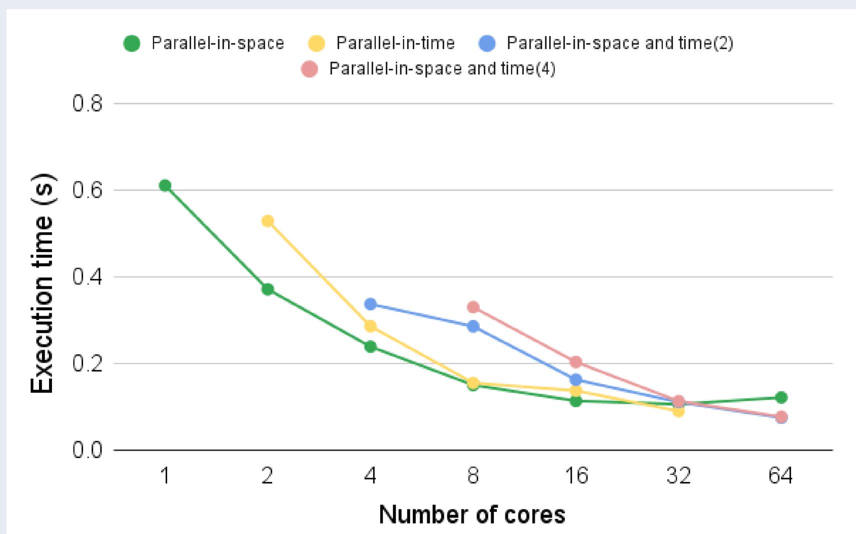


Figure 3:Execution time using parallel-in-space and parallel-in-time (proposed method). The number in the parentheses is the number of processors in time dimension.

2D CFD

We solve the following Navier-Stokes Equations to simulate a compressible fluid flow.

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u\vec{i} + \rho v\vec{j}) = 0 \\ \frac{\partial (\rho u)}{\partial t} + \nabla \cdot ((\rho u^2 + p)\vec{i} + \rho uv\vec{j}) = \nabla \cdot (\tau_{xx}\vec{i} + \tau_{xy}\vec{j}) \\ \frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho uv\vec{i} + (\rho v^2 + p)\vec{j}) = \nabla \cdot (\tau_{xy}\vec{i} + \tau_{yy}\vec{j}) \\ \frac{\partial E}{\partial t} + \nabla \cdot ((Eu + pu)\vec{i} + (Ev + pv)\vec{j}) = \\ \quad \nabla \cdot ((u\tau_{xx} + v\tau_{xy} - q_x)\vec{i} + (u\tau_{xy} + v\tau_{yy} - q_y)\vec{j}) \end{cases}$$

There are 5 parameters: density ρ, x-velocity u, y-velocity v, energy E, and pressure p. Pressure p can be derived from the following equation.

$$p = (\gamma - 1) * (E - \frac{1}{2}\rho(\mathbf{u}^2))$$

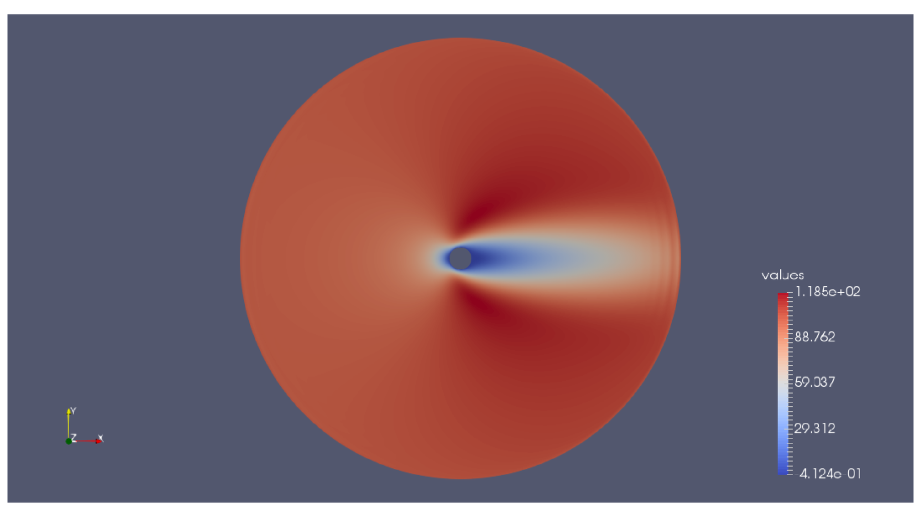


Figure 4:We simulate air flow around a cylinder with a speed of 0.3 Mach.

The four coupled equations are solved by finite volume method with Lax-Wendroff scheme (explicit). Restriction and Prolongation between fine and coarse layers are calculated by bicubic interpolation.

$$\int_{\Omega_N} \left(\frac{\partial U}{\partial t}\right) d\Omega + \int_{\Omega_N} \left(\nabla \cdot \vec{F} - \nabla \cdot \vec{R}\right) d\Omega = 0$$

$$\left(\frac{\partial U}{\partial t}\right)_N = -\frac{1}{\Omega_N} \int_{\partial \Omega_N} (\vec{F} - \vec{R}) \cdot \hat{n} dS$$

Result (CFD)

The experiment constructs a PinT method with 4 layers. The PinT execution time is compared to the execution of pure spaital parallelization.

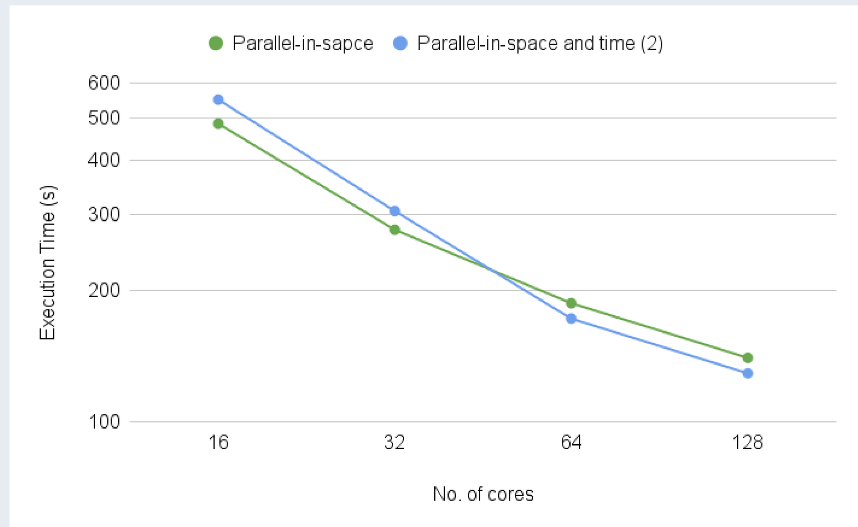


Figure 5:Comparison of execution time using only spatial parallelization and parallel-in-space/time.

Conclusion

This work propose a parallel-in-time method for explicit schemes. This work shows that parallelizing in both space and time with our method is more efficient than pure spatial computation, without requiring extremely large number of processors.

Future Work

We are trying to apply Cascadic Parareal onto supersonic flow simulation with adaptive mesh refinement. To solve the convergence problem with shock waves.

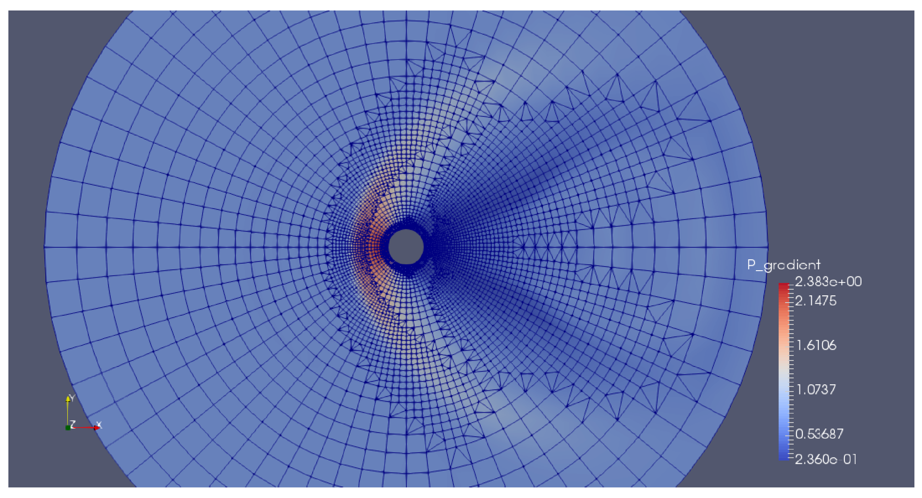


Figure 6:Apply adaptive mesh refinement for shock wave simulations can improve the computational accuracy.

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