# IC(p) preconditioning for large symmetric linear equations

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## 1. INTRODUCTION

Iterative methods are often used to solve large-scale linear equations such as high frequency electromagnetic field analysis [1]. However, as the matrix size becomes larger, convergence of iterative methods such as the Krylov subspace method becomes worse, and it takes a lot of time to solve equations. The preconditioning method such as Incomplete Cholesky (IC(p)) preconditioning [2] with fill-in is used to improve the convergence of iterative methods, where p is the level of fill-in. While increasing the fill-in level generally improves convergence, the number of non-zero elements in the preconditioning matrix increases by a factor of several, sometimes more than ten, for each additional level, which can exhaust memory in huge problems.

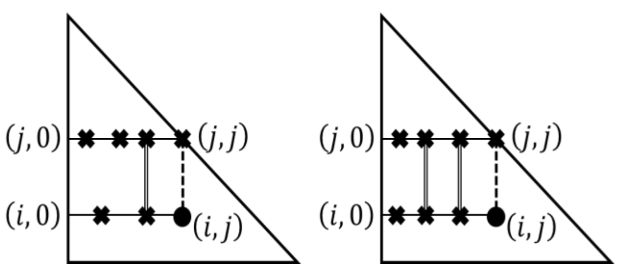
Therefore, in this study, we present a method that is able to control the level of fill-in independent of the matrix value and show the results for electromagnetic field problems.

## 2. FILL-IN

Figure 1 shows how to determine the position of additional non-zero elements in the preconditioning matrix. In this study, we propose a method, called IC(0.5), which can be regarded as an intermediate method between IC(0) and IC(1).

In IC(1), the additional non-zero elements are determined as shown in Fig. 1(a); the element in $(i, j)$ becomes additional non-zero if there are any non-zero elements in rows $i$ and $j$ that have the same column.

On the other hand, in our IC(0.5), additional non-zero elements are determined as shown in Fig. 1(b); the element in $(i, j)$ becomes additional non-zero if *two or more* non-zero elements in rows $i$ and $j$ have the same column. In this method, we can add non-zero elements that have large effects to convergence (that means to improve convergence well) and reject others.




**(a) Fill-in with IC(1)** **(b)Fill-in with IC(0.5)**

**Figure 1: How to determine the additional non-zero element positions.**

## 3. NUMERICAL EXPERIMENTS

All calculations were performed using an Intel CPU Core i9-9900 (3.60 GHz) processor with 32 GB memory, the gcc 11.4.0 compiler, and the "-O3 -mfma" optimization flag. As a high frequency electromagnetic field problem, let us consider a wave equation having an electric field derived from Maxwell's equation. We constructed problems with 160,013 and 979,464 degrees of freedom by ADVENTURE_Magnetic [3]. The initial solution of the equation $Ax = b$ was set to $x = 0$ and the convergence tolerance was set to $10^{-9}$.

Table 1 shows the number of non-zero elements in the preconditioning matrix at different fill-in levels. The proposed IC(0.5) successfully increased the number of non-zero elements than IC(0) and reduced the number of non-zero elements by about half compared to IC(1).

Table 2 shows the iteration counts and calculation time. As can be seen in the table, our proposed method succeeded in reducing the calculation time by up to 19% compared to IC(1).

**Table 1. The Number of non-zero elements in the preconditioning matrix.**

| size | IC(0) | IC(0.5) | IC(1) |
|---|---|---|---|
| 160,013 | 1,343,678 | 1,763,200 | 3,516,247 |
| 979,464 | 8,387,104 | 11,080,164 | 22,268,605 |

**Table 2. Performance evaluation.**

| | IC(0) | | IC(0.5) | | IC(1) | |
|---|---|---|---|---|---|---|
| size | iter. | time [s] | iter. | time [s] | iter. | time [s] |
| 160,013 | 9,491 | 202 | 8,026 | 197 | 5,264 | 199 |
| 979,464 | 8,199 | 1,106 | 6,519 | 1,015 | 5,312 | 1,249 |

## 4. CONCLUSION

In this study, we proposed IC(p) preconditioning method that can control the level of p. Moreover, we implemented an iterative method with different fill-in levels and evaluate the performance for complex symmetric matrices that appear in high-frequency electromagnetic field analysis. Our proposed method succeeded in controlling the non-zero elements, and reducing the calculation time compared to IC(0) and IC(1).

In future work, we investigate the effectiveness of the method with other problems and consider better fill-in methods.

## REFERENCES

[1] Amane Takei, Shinobu Yoshimura, Hiroshi Kanayama. 2008. Large-Scale Parallel Finite Element Analyses of High Frequency Electromagnetic Field in Commuter Trains. Comput. Modeling Engrg. Sci., 31 (Jul. 2008), pp.13-24.
[2] Meijerink J. A. and van der Vorst H. A. 1977. An Iterative Solution Method for Linear Systems of Which the Coefficient Matrix is a Symmetric M-Matrix. Mathematics of Computation, 31, 137 (Jan 1977), pp. 148-162.
[3] ADVENTURE Project, https://adventure.sys.t.u-tokyo.ac.jp/